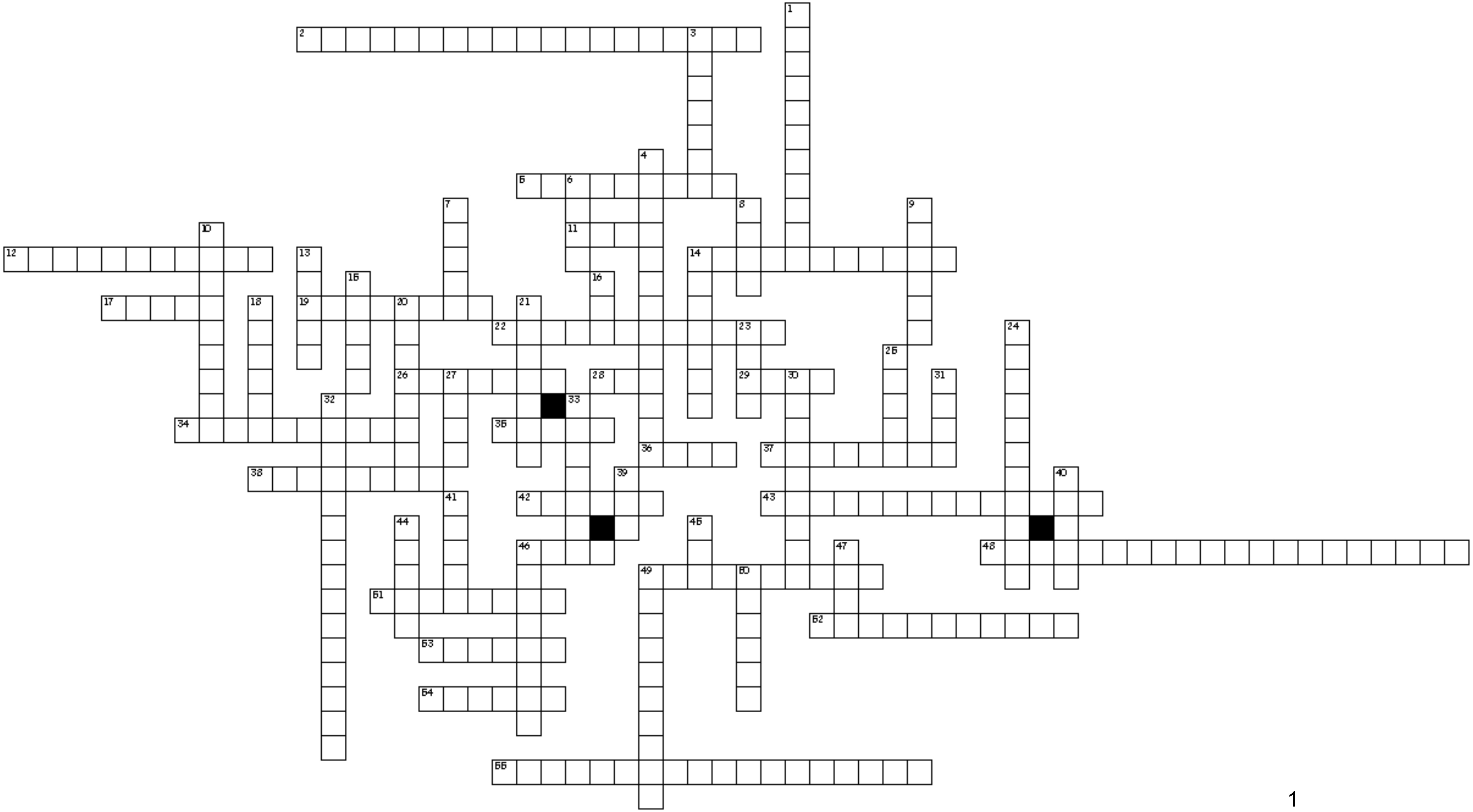
Infinity activities

- Infinity crosswords
- Fill in the gaps
- Conceptual map

Infinity crosswords



Across

2. The cardinality of the real numbers is the next bigger cardinality after the cardinality of the natural numbers.
5. The mathematician who discovered the existence of unprovable statements like the continuum hypothesis
11. Coming after all others in time or order; final.
12. The number of elements in a set, whether the set is finite or infinite. Note: Not all infinite sets have the same cardinality.
14. A positive integer which has only 1 and the number itself as factors. For example, 2, 3, 5, 7, 11, 13, etc.
17. Two times; on two occasions.
19. An attribute, quality, or characteristic of something.
22. The German mathematician who tried to explain some of the properties of infinity using a hotel with infinitely many rooms.
26. Extend over so as to cover partly. You can say, for example: ‘the curtains overlap at the centre when closed’.
28. A group of numbers, variables, geometric figures, or just about anything.
29. On one occasion or for one time only.
34. A statement that cannot be proved.
35. The result of raising a base to an exponent.
36. Coming immediately after the previous one.
37. All positive and negative whole numbers (including zero). That is, the set {... , −3, −2, −1, 0, 1, 2, 3, ...}.
38. A "number" which indicates a quantity, size, or magnitude that is larger than any real number.
42. An arithmetical value, expressed by a word, symbol, or figure, representing a particular quantity and used in counting and making calculations.
43. The numbers used for counting. That is, the numbers 1, 2, 3, 4, etc.
46. Containing or holding as much or as many as possible; having no empty space.
48. Proving a conjecture by assuming that the conjecture is false. If this assumption leads to a contradiction, the original conjecture must have been true.
49. The part of a conditional after If and before then.
51. Straight up and down. As an example, we can consider a wall.
52. The first mathematician to realize that there are different kinds, different sizes of infinity.
53. Describes a set which does not have an infinite number of elements. That is, a set which can have its elements counted using natural numbers. Formally, a set is finite if its cardinality is a natural number.
54. A mark or character used as a conventional representation of an object, function, or process, e.g. the letter or letters standing for a chemical element or a character in musical notation.
55. Writing an integer as a product of powers of prime numbers.

Down

1. Describes a set which contains more elements than the set of integers. Formally, an uncountably infinite set is an infinite set that cannot have its elements put into one-to-one correspondence with the set of integers. For example, the set of real numbers is uncountably infinite.
3. Think or assume that something is true or probable.
4. A logical incompatibility between two or more propositions.
6. One of a set of explicit or understood regulations or principles governing a particular area of activity.
7. A person staying at a hotel.
8. A network of lines that cross each other to form a series of squares or rectangles.
9. Quick to understand, learn, and apply ideas; intelligent.
10. A function between the elements of two sets, where every element of one set is paired with exactly one element of the other set, and every element of the other set is paired with exactly one element of the first set. There are no unpaired elements.
13. Containing nothing; not filled or occupied.
14. An inquiry starting from given conditions to investigate or demonstrate a fact, result, or law.
15. Find all solutions to an equation, inequality, or a system of equations and/or inequalities.
16. An integer that is not a multiple of 2.
18. Any integer which divides evenly into a given integer.
20. x in the expression a^x.
21. A seemingly absurd or contradictory statement or proposition which when investigated may prove to be well founded or true.
23. A part or division of a building enclosed by walls, floor, and ceiling.
24. All numbers on the number line. This includes (but is not limited to) positives and negatives, integers and rational numbers, square roots, cube roots , π (pi), etc.
25. A continuous area which is free, available, or unoccupied.
27. An integer that is a multiple of 2.
30. Describes the cardinality of a countably infinite set. For example, the set of natural numbers is countably infinite. Aleph null ($\aleph_0$) is the symbol for this.
31. A mathematical operation of addition.
32. All positive and negative fractions, including integers and so-called improper fractions. Formally, rational numbers are the set of all real numbers that can be written as a ratio of integers with nonzero denominator.
33. Relating to or denoting a system of numbers and arithmetic based on the number ten, tenth parts, and powers of ten.
39. Produced, introduced, or discovered recently or now for the first time; not existing before.
40. A convincing demonstration that some mathematical statement is necessarily true.
41. Any of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 used to write numbers.
44. An establishment providing accommodation, meals, and other services for travellers and tourists, by the night.
45. A break or hole in an object or between two objects.
46. A numerical quantity that is not a whole number.
47. A thing’s dimensions or magnitude; how big something is.
49. Perfectly flat and level. For example, we can think to the horizon. So is the floor.
50. A contemplative and rational type of abstract or generalizing thinking, or the results of such thinking.

Fill in the gaps

Fill in the gaps with the words listed below: (pay attention: not all the words are necessary!)

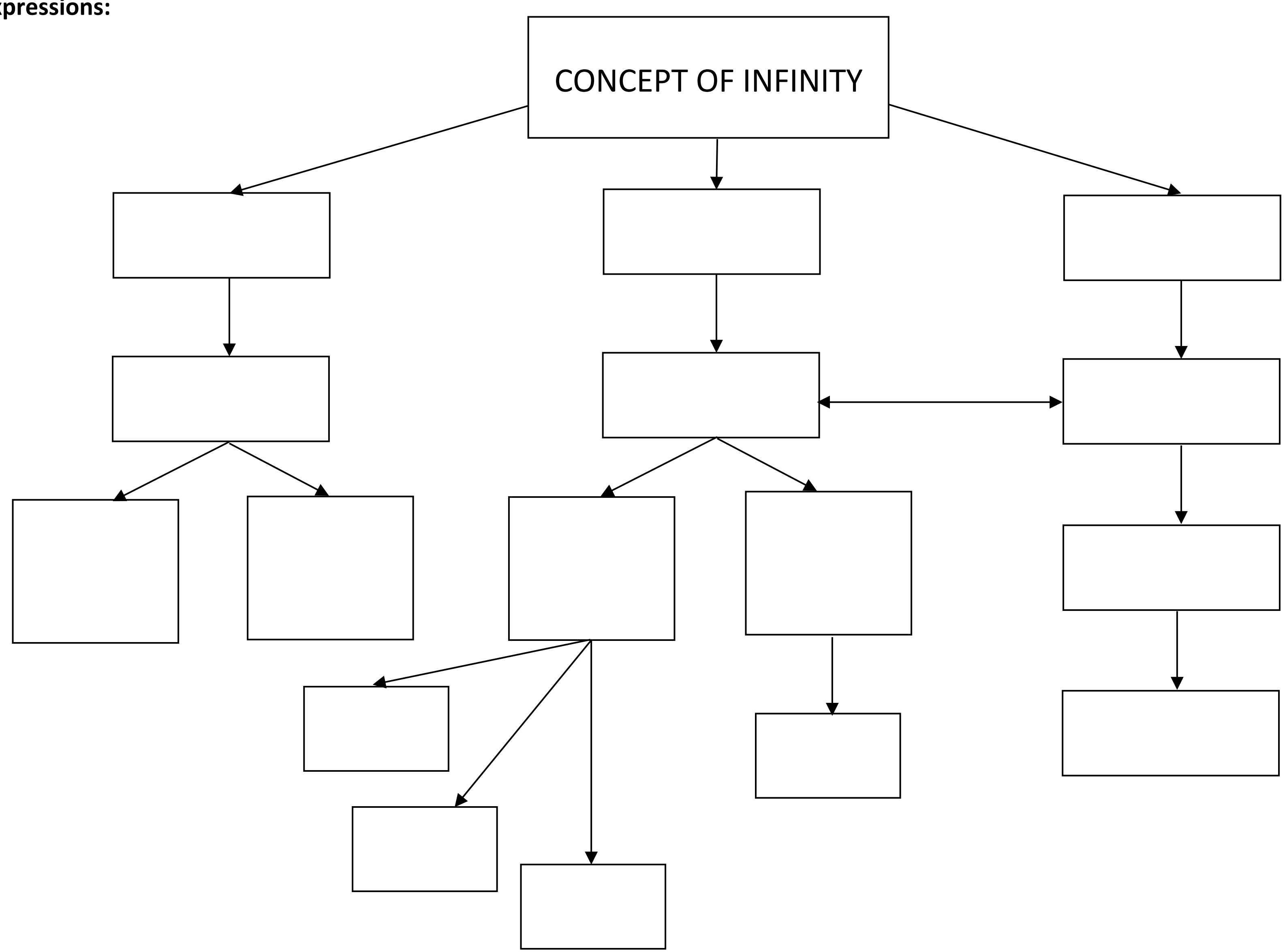
cardinality - integers – passengers - hotel – **Set Theory** – seats - kind - real numbers – math - empty - guests – paired up - **countable** – size - **Continuum Hypothesis** – full - numbers – mathematician – room –move –cardinality - infinity

Infinity is without doubt one of the hardest mathematical concepts to understand.
The mathematician David Hilbert tried to explain some of the properties of _______ using a hotel with infinitely many rooms. We could imagine that it consists of just a single, never-ending corridor, with the rooms numbered 1, 2, 3, 4 and so on. On the day we arrive at this _____, every single room is full. In any other hotel, the reception would have to turn us away.
Here, instead, once we ask for a _____, the hotel asks all the guests to move up into the next room. Since there is no last room in the hotel, every guest will get a new room.
Now the first room is _____, and we can move in. The Hilbert doesn't have a last room!
This idea can be extended: thus even if the hotel is ____ it can still offer rooms to *any* finite number of new guests!
Let us suppose that the hotel is full again, and that *infinitely* many new guests arrive. Now the previous method won't work anymore: you can't ask the ______ to move up infinitely many rooms. Instead the hotel asks the guests to move into the room with *twice* the number of their current room.
Finally let us suppose that a whole bus company arrives at the hotel: they have infinitely many buses, each with infinitely many _____. Even if it's more difficult, we can also move all those infinitely times infinitely many passengers into this already full hotel.
If we want to go even further, we have to develop a new language that is called ___________.
A set is simply a collection of any objects. We say that two sets have the same size if we can build a **bijection** pairing up their elements so that no element is left out or connected to two elements. This applies to finite sets as well as infinite sets, and we say that a set is _________ if it can be paired up with the natural numbers.
We can pair up the natural numbers with *all* integers: it seems that there are as many positive ________ as there are integers in total.
We can even pair up the natural _________ with the positive rational numbers.
It now seems quite likely that we must be able to pair up any number set with the natural numbers, but this is not the case. One example are the real numbers. Of course there are infinitely many real numbers, but this infinity is of a "bigger _____" than before, and we say that the real numbers are *uncountable*. To prove that the real numbers can't be ___________ with the natural numbers we use **proof by contradiction**.
Georg Cantor was the first __________ to realize that there are different *sizes* of infinity. He called these different sizes **cardinalities**.
We have shown above that $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ all have the same cardinality, which is called $\aleph_0$. We have also shown that the real numbers $\mathbb{R}$ have a bigger __________.
Georg Cantor asked whether the cardinality of $\mathbb{R}$ is the next bigger cardinality after $\aleph_0$. This is called the ________________________.

Conceptual map

Fill in the blocks with suitable expressions:

- Set Theory
- Countable sets
- Rational numbers ($\mathbb{Q}$)
- Real numbers ($\mathbb{R}$)
- Cardinality
- Georg Cantor
- Full hotel but rooms available for new infinite guests
- Bijection
- Integers ($\mathbb{Z}$)
- Different sizes of infinity
- David Hilbert
- Continuun Hypothesis
- Uncountable sets
- Natural numbers ($\mathbb{N}$)
- Hotel with infinitely many rooms
- Full hotel but rooms available for new finite guests



Activity support

Introduction

David Hilbert (1862–1943)

Infinity is without doubt one of the hardest mathematical concepts to understand. Many ideas that we find intuitive when working with normal numbers don't work anymore, and instead there are countless apparent paradoxes. Is there a largest number? Is there anything bigger than infinity? What is infinity plus one? What is infinity plus infinity?

The symbol for infinity is ∞, a horizontal 8. It was invented by John Wallis (1616–1703) who could have derived it from the Roman numeral M for 1000. But one thing we know for certain: infinity is a lot bigger than 1000.

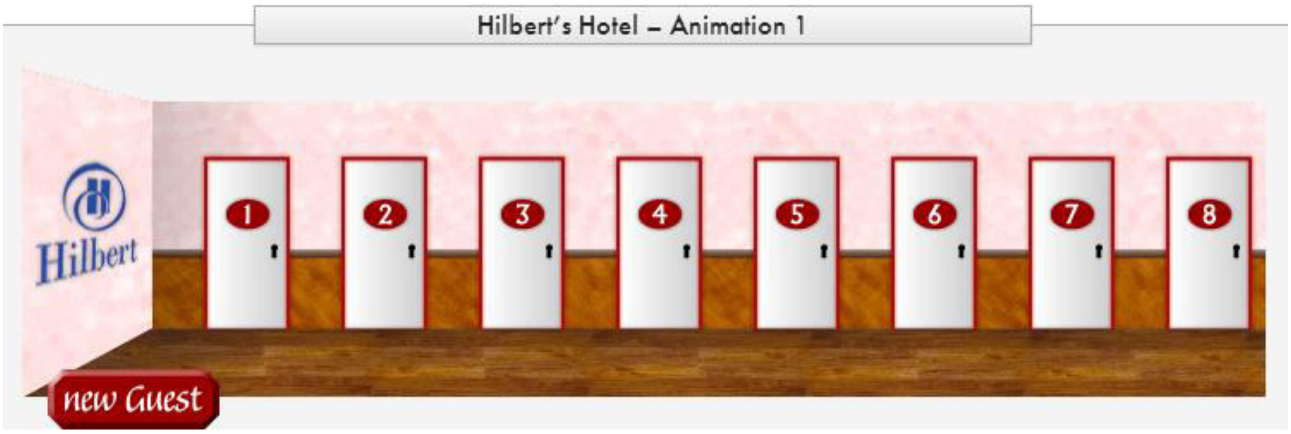
The German mathematician David Hilbert (1862–1943) tried to explain some of the properties of infinity using a hotel with infinitely many rooms. Hilbert is also well-known for presenting a list of 23 mathematical problems which he considered the most important ones at the start of the 20th century. Some of these problems, like the Riemann Hypothesis, are *still* unsolved.

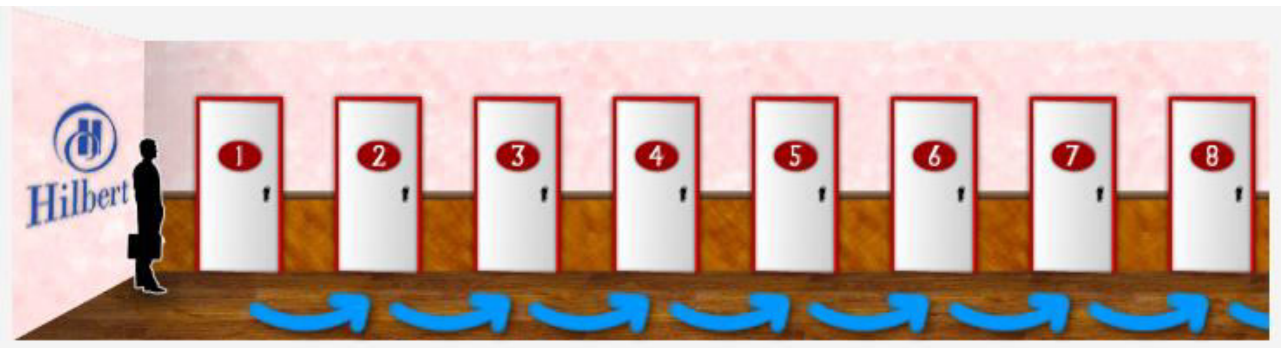
Hilbert's Hotel

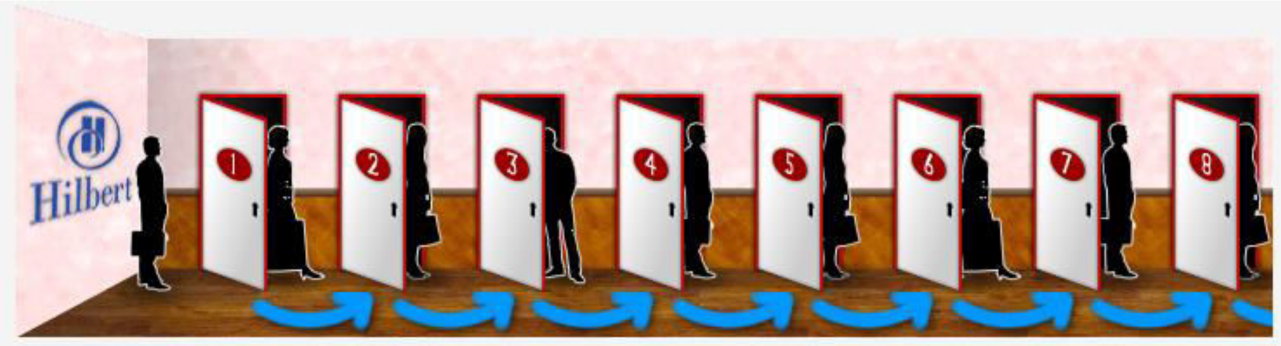
The *Hilbert* is the largest hotel in the world: it has infinitely many rooms. We could imagine that it consists of just a single, never-ending corridor, with the rooms numbered 1, 2, 3, 4 and so on. On the day we arrive at the Hilbert, every single room in the Hotel is full. (Business is going very well indeed…)

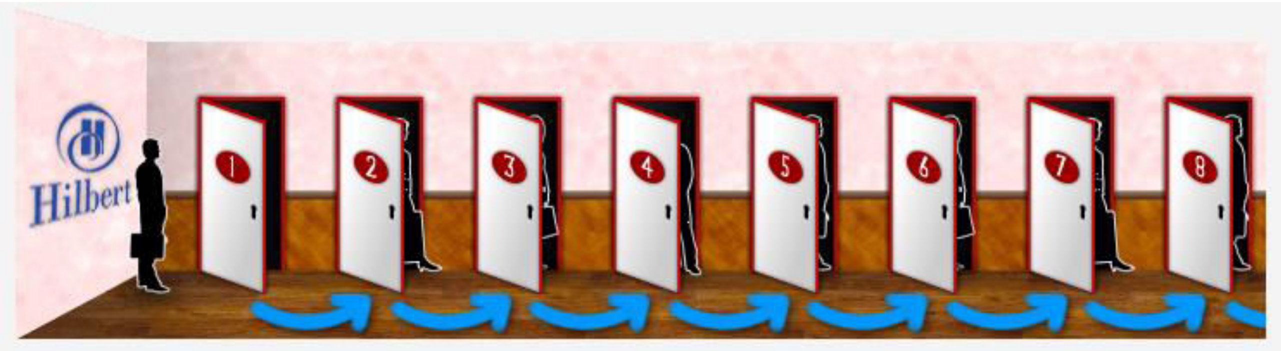
In any other Hotel, the reception would have to turn us away: if all the rooms are full there simply isn't space for us. Very surprisingly, this is not the case in the Hilbert…

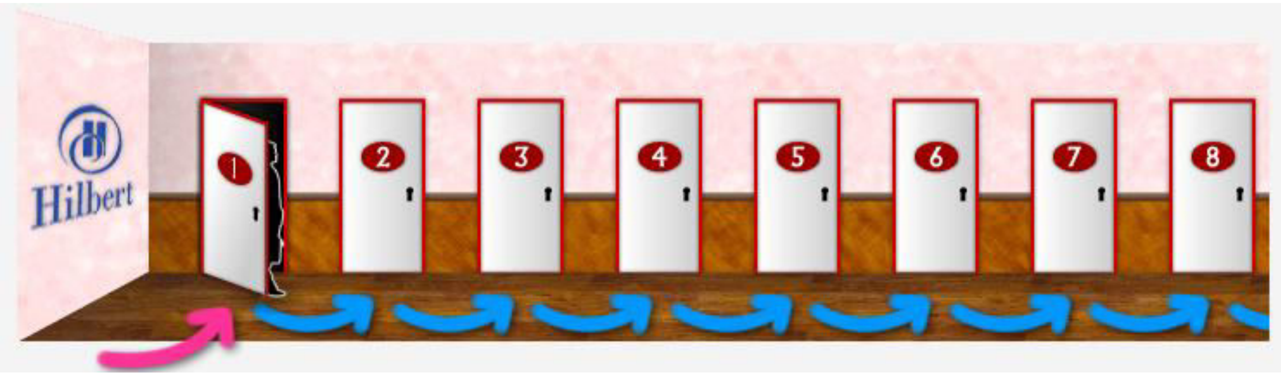
Once we arrive and ask for a room, the hotel makes a loudspeaker announcement that can be heard everywhere in the hotel: they ask all their guests to move up into the next room. Now the guest in room 1 moves into room 2, the guest in room 2 moves into room 3, and the guest in any room n moves into room $n + 1$. Since there is no last room in the hotel, every guest will get a new room.











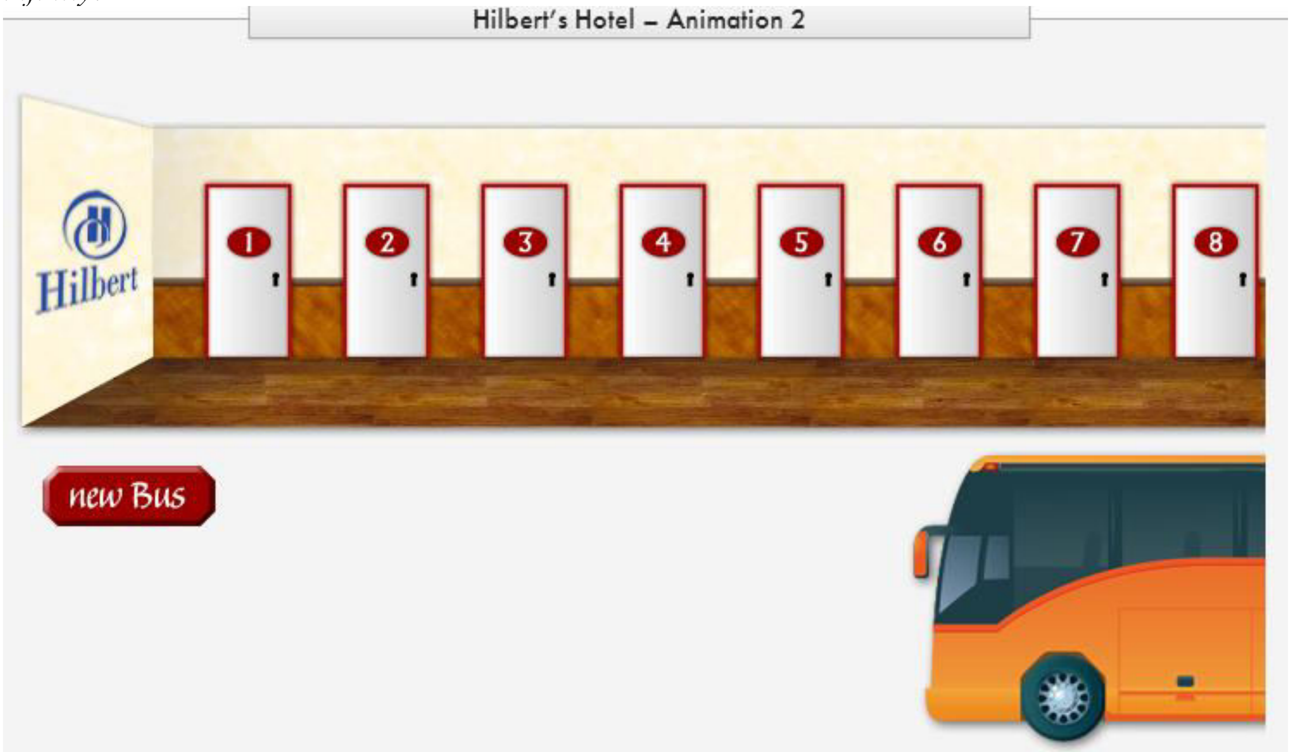
Now the first room is empty, and we can move in. Of course this wouldn't have worked in any finite hotel. We could still ask all guests to move up into the next room, but the person in the last room would be left without a room. The Hilbert doesn't have a last room, and *infinity plus one is still infinity*.

This idea can be extended: If 10 new guests arrive, the reception just asks all guests to move up 10 rooms. If 100 new guests arrive, all current guests have to move up 100 rooms – and similarly for any other number. Thus even if the hotel is full it can still offer rooms to *any* number of new guests!

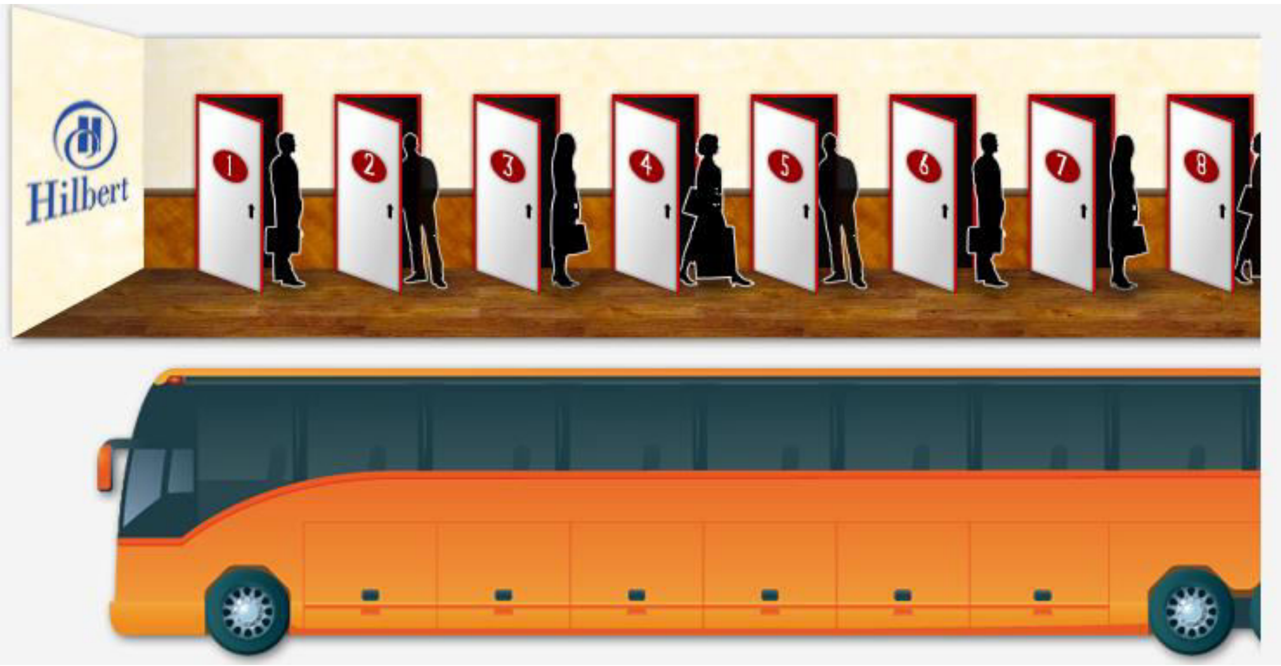
Let us suppose that the hotel is full again, and that *infinitely* many new guests arrive. Now the previous method won't work anymore: you can't ask the guests to move up infinitely many rooms – they would never arrive in their new room.

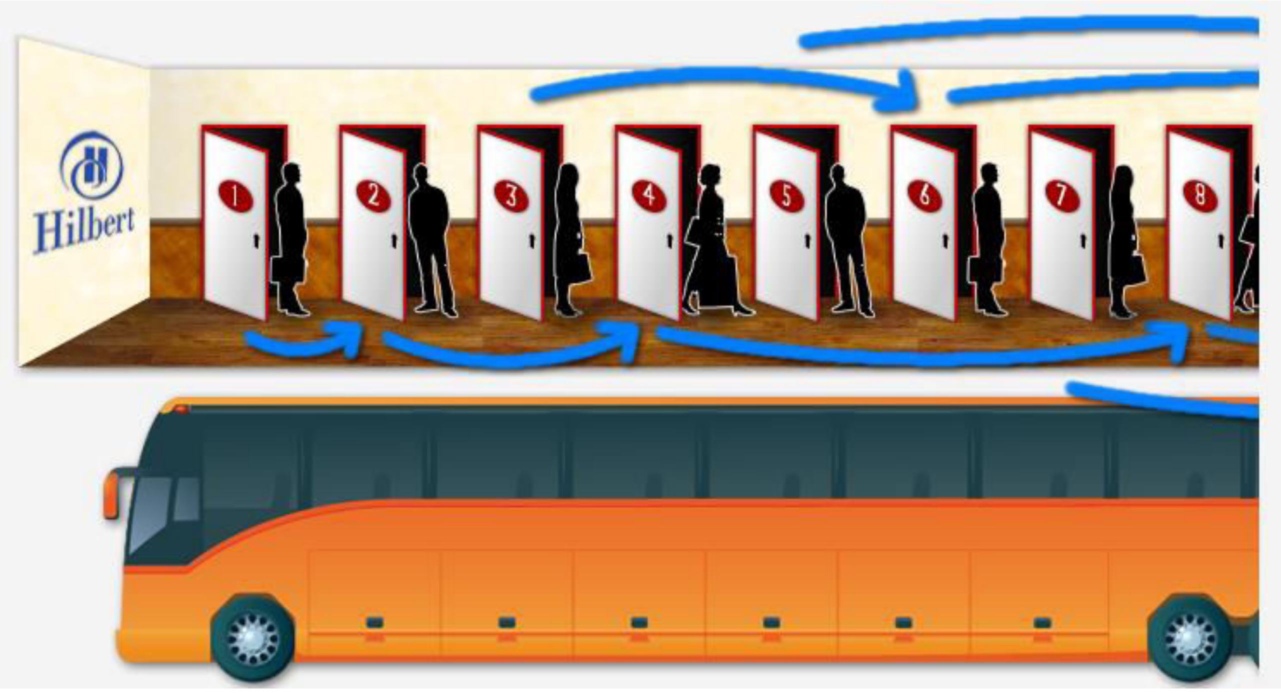
Instead the hotel does something very clever: they ask their guests to move into the room with *twice* the number of their current room. Thus the guest in room 1 will move into room 2, the guest in room 2 will move into room 4, and so on. The guest in room n will move into room $2n$.

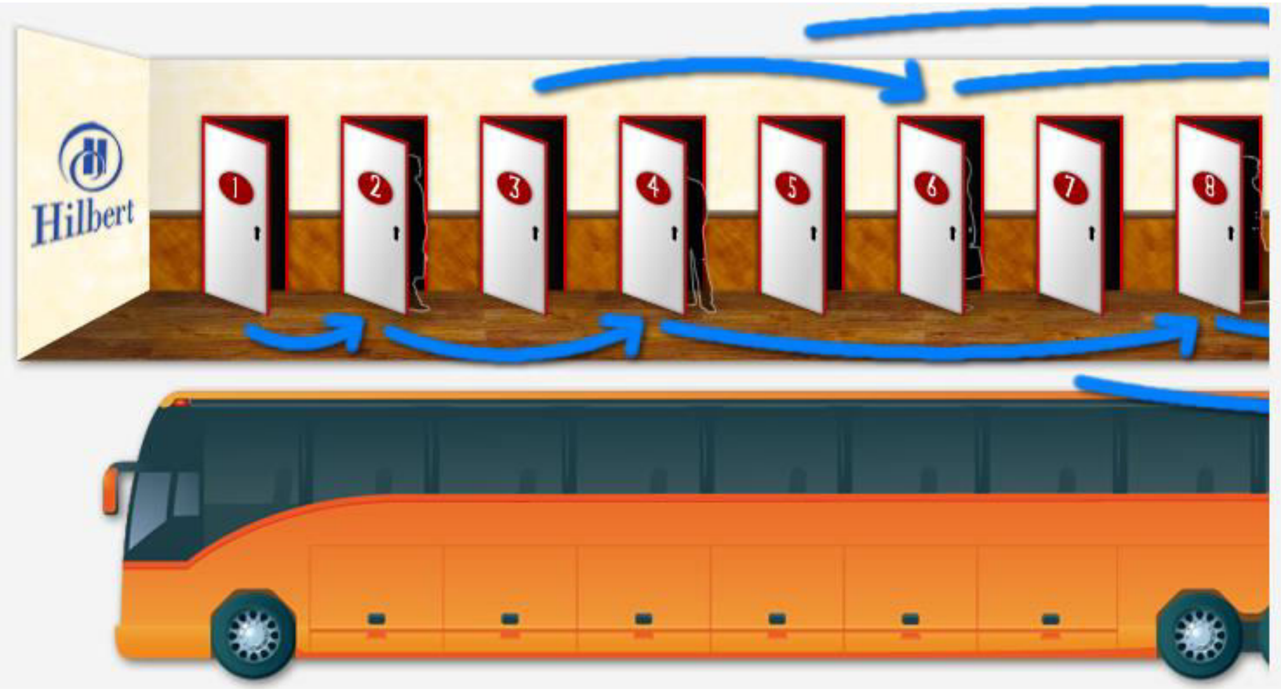
Notice that now all the odd numbered rooms are empty. Since there are infinitely many odd numbers, there is enough space for the infinitely many new guests. *Infinity plus infinity is still infinity*.

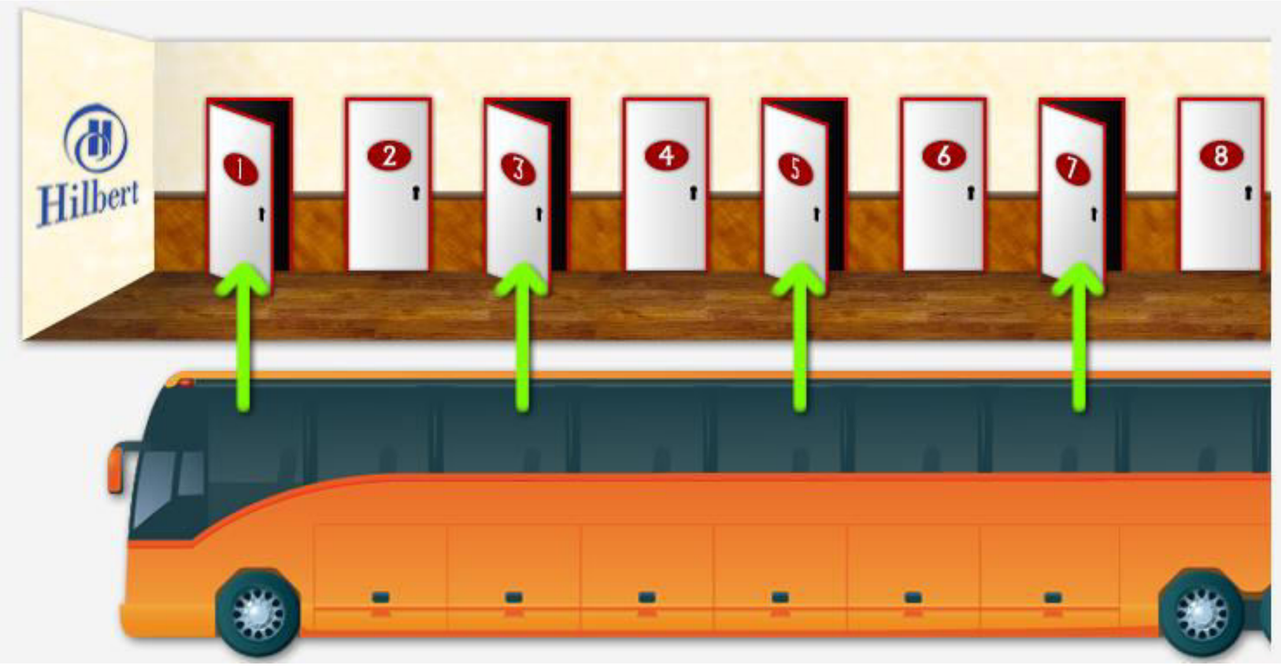


An infinite bus arrives at the Hilbert. The hotel is completely full, yet everybody gets a room.




Hilbert
1
2
3
4
5
6
7
8


Hilbert
1
2
3
4
5
6
7
8



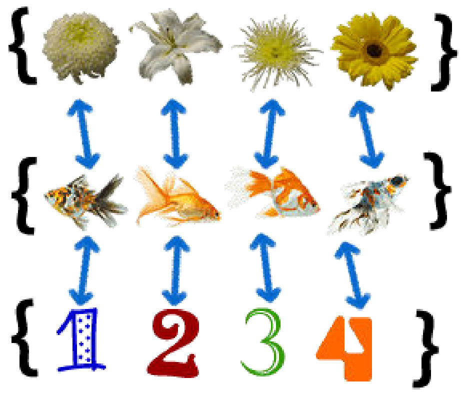
Finally let us suppose that a whole bus company arrives at the hotel: they have infinitely many buses, each with infinitely many seats. Can we also move all those infinitely times infinitely many passengers into one hotel?

It turns out that we can, using a simple rule: the passenger sitting on seat number i in bus number j gets room $2^i\ 3^j$ in the hotel. Here *i* and *j* can be any numbers, for example the passenger on seat 5 in bus 2 gets room $2^5\ 3^2 = 32 \times 9 = 288$.

Of course we have to check that we don't happen to give two passengers the same room. This is because 2 and 3 are prime numbers and because prime factorisation is unique – meaning that if two numbers are different, they have a different prime factorisation. But $2^i\ 3^j$ is the prime factorisation of the room numbers we assign to the customers, so different passengers will always get a different room.

In addition, all rooms which have prime factors other than 2 and 3, for example rooms 5, 7, 10, and so on, remain empty. As we walk down the infinite hotel corridor, only very few of the rooms will be used. This is useful in case an infinite bus arrives in the middle of the night and you don't want to wake everybody up.

Countability

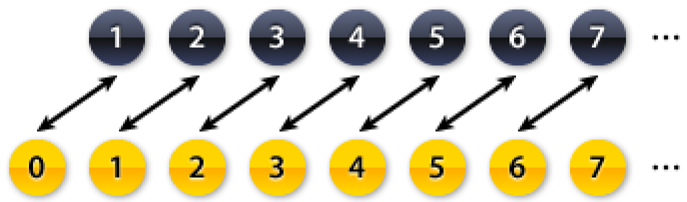


If we want to go even further, stretching our imagination and the meaning of infinity, we can't use hotels. We have to develop a new language to express our ideas. This language is called **Set Theory**.

A set is simply a collection of any objects – fruit, buildings, or numbers – often written inside {} brackets. We say that two sets have the same size if we can pair up their elements so that no element is left out or connected to two elements. Such a pairing is called a **bijection**.

This applies to finite sets as well as infinite sets, and we say that a set is **countable** if it can be paired up with the natural numbers $\mathbb{N}= \{1, 2, 3, 4, 5, \ldots\}$. In that case, we can write down all elements in a numbered list.

For example, we can add one more number to $\mathbb{N}$, say 0, and see whether we can pair up the two sets:

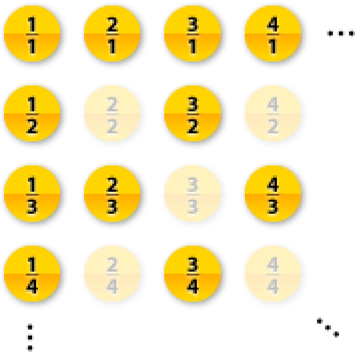


Since the black and the yellow set can be paired up without gaps or overlaps, they must have the same size. Infinity plus one is still infinity. This is precisely the same principle as in Hilbert's Hotel above, where we paired up the infinitely many room numbers with the infinitely many guests.

We can do even better: let us try to pair up the positive integers $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$ with *all* integers (these are written as $\mathbb{Z} = \{\ldots,-3, -2, -1, 0, 1, 2, 3, \ldots\}$). Now we have to jump back and forth between the positive and negative ends of the number line and we can be sure, counting the integers ($\mathbb{Z}$), not to forget any element.

Infinity plus infinity is still infinity, and it seems that there are as many positive integers as there are integers in total.

We can even try to pair up the positive integers ℕ with the positive rational numbers ℚ. Rational numbers are all fractions, such as 1/2, 7/40, 13/17 and so on. To do this, we first write down all fractions in an infinite grid where the columns are the numerators and rows the denominator. Some fractions appear several times: 1/2, for example, is the same as 2/4 or 3/6 and so. Therefore we delete all the duplicates and only keep the smallest version:



Now we follow a path through this square, and every time we see a valid rational number, we pair it up with the next integer: once again, we can be sure not to forget any rational number.

And this is really quite surprising: remember that there are infinitely many rational numbers between 0 and 1. (Actually, there are infinitely many rationals between *any* two distinct numbers, no matter how close they are.) But since we can pair up all these rationals with the natural numbers, there must be the same number. Infinity times infinity is still infinity.

Cantor's Diagonal

It now seems quite likely that we must be able to pair up any number set with the natural numbers, but this is not the case. One example are the real numbers ℝ, which include all the fractions but also numbers like Pi, *e* or √2. Of course there are infinitely many real numbers, but this infinity is of a "bigger kind" than before, and we say that the real numbers are *uncountable*.

To prove that the real numbers can't be paired up with the natural numbers, we use **proof by contradiction**. We start by *assuming* that the real numbers *were* countable. Then we deduce something we know is wrong: a contradiction. This means that our initial assumptions must have been wrong, i.e. that the real numbers are *not*countable.

All real numbers can be expressed as a decimal. The decimal expansion could go on forever, like 0.3333333…, or stop, in which case we can add 0s like in 0.12300000… Let us suppose that the real numbers *were* countable. Then we could write down *all* real numbers in a list. Below, every row is a new number, and the columns are the decimal expansions of the numbers:

$$
\begin{array}{lllllll}
x_1 & . & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & \dots \\
x_2 & . & x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & \dots \\
x_3 & . & x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & \dots \\
x_4 & . & x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & \dots
\end{array}
$$

…

We now construct a new number consisting of all the elements along the diagonal:

$$0 \;.\; x_{11} \quad x_{22} \quad x_{33} \quad x_{44} \quad x_{55} \quad \dots$$

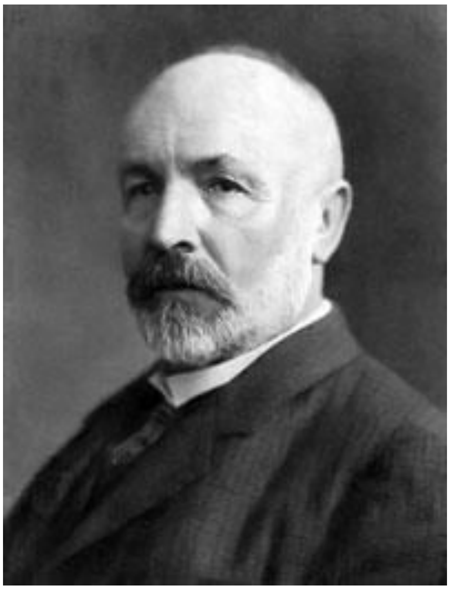
Now we change all the digits in this new number. For example, we could make every digit a 1, except the 1s, which become 2s:

$$0 \;.\; y_{11} \quad y_{22} \quad y_{33} \quad y_{44} \quad y_{55} \quad \dots$$

This is again a real number. Notice, however, that it is not in our list of all real numbers above: it can't be the first number, since the x_{11} digit doesn't match. It can't be the second number since the x_{22} digit doesn't match. And so on – $0.y_{11}y_{22}y_{33}\dots$ is a completely new real number not in our initial list.

But this is a contradiction, since we started by assuming that *all* real numbers were in our list. Therefore this assumption must have been wrong, so we can't write down all real numbers in a list. Therefore the real numbers are not countable. They are a bigger kind of infinity.

The Continuum Hypothesis



Georg Cantor (1845–1918)

Georg Cantor (1845–1918) was the first mathematician to realize that there are different *kinds*, different *sizes* of infinity. He called these different sizes **cardinalities**. Cantor viewed his mathematical discoveries in a very philosophical way and believed that his study of infinity could explain the absolute infinity of God. During his life, his research was dismissed as ‘shocking’ and ‘corrupting’, but it is fundamental to modern mathematical logic.

We have shown above that $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ all have the same cardinality, which is called Aleph 0 and written as $\aleph_0$ (Aleph is the Hebrew letter A). We have also shown that the real numbers $\mathbb{R}$ have a bigger cardinality.

One question which Georg Cantor asked is whether the cardinality of $\mathbb{R}$ is the next bigger cardinality after $\aleph_0$ (in the same way that 1 is the next bigger integer after 0). This is called the **Continuum Hypothesis**. Cantor spent many years of his life trying to prove it – but he was unsuccessful.

Only much later, in 1963, mathematicians discovered something very curious: The continuum hypothesis could be true or false, and it is impossible to prove it either way. It basically means that you can decide for yourself whether you *want* it to be true or not. The existence of unprovable statements like the continuum hypothesis was discovered by Kurt Gödel.

Sitography:

http://world.mathigon.org/Infinity