Insiemi infiniti e cardinalità - CLIL

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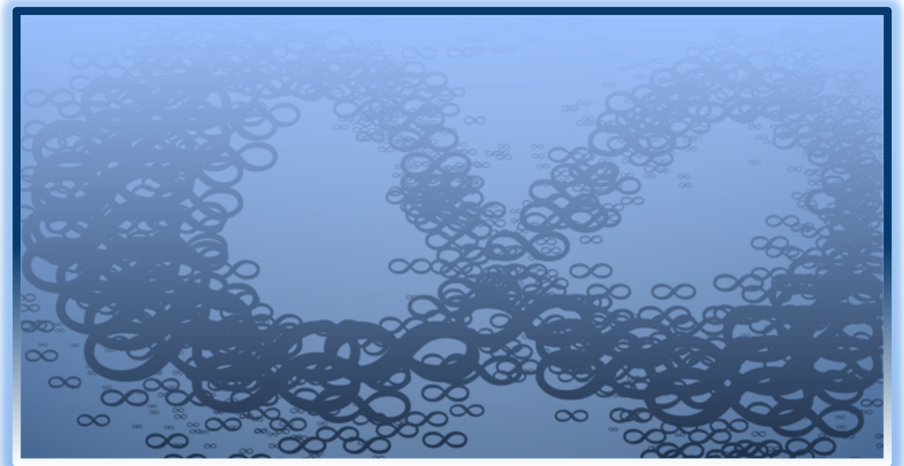
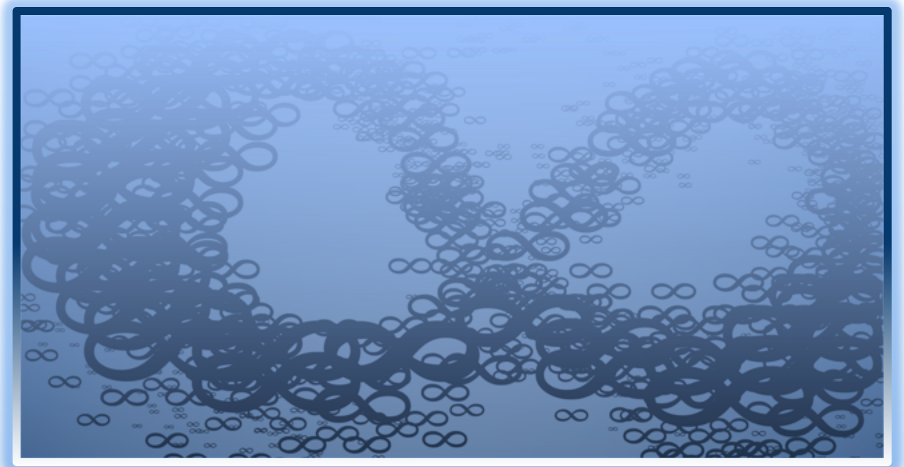
INTRODUCTION



- **Infinity** is without doubt one of the **hardest mathematical concepts** to understand. Many ideas that we find intuitive when working with normal numbers don’t work anymore, and instead there are countless apparent **paradoxes**.
- Is there anything **bigger than infinity**? What is **infinity plus one**? What is **infinity plus infinity**?
- The German mathematician **David Hilbert** tried to explain some of the properties of infinity using a hotel with infinitely many rooms.

HILBERT'S HOTEL



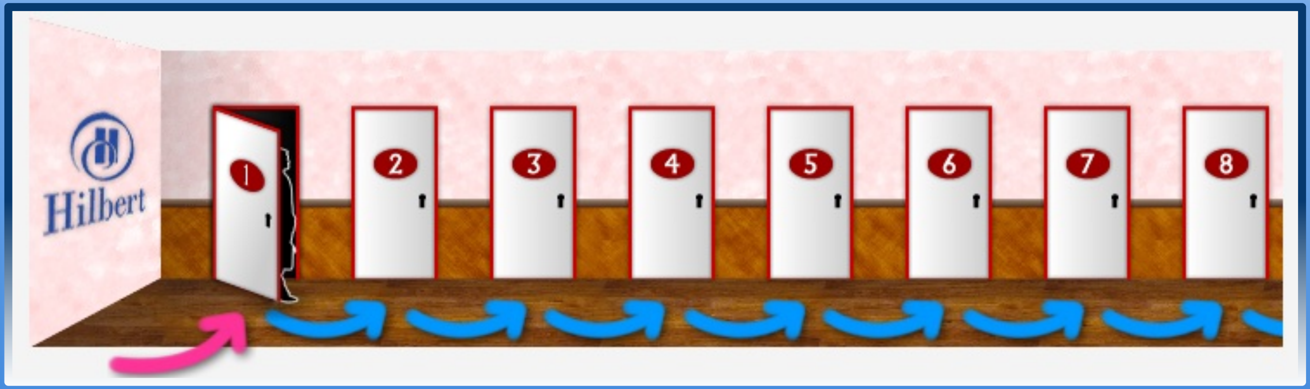
- The ***Hilbert*** is the largest hotel in the world: it has **infinitely many rooms**. We could imagine that it consists of just a single, never-ending corridor, with the rooms numbered 1, 2, 3, 4 and so on.

- On the day we arrive at the Hilbert, every single room in the Hotel is **full**.

- In any other Hotel, the reception would have to **turn us away**. Very surprisingly, **this is not the case** in the Hilbert…

- Once we arrive and ask for a room, the hotel makes a **loudspeaker announcement** that can be heard everywhere in the hotel: **they ask all their guests to move up into the next room**.

- The guest in room 1 moves into room 2, the guest in room 2 moves into room 3, and the guest in any room n moves into room n + 1. Since there is no last room in the hotel, **every guest will get a new room.**



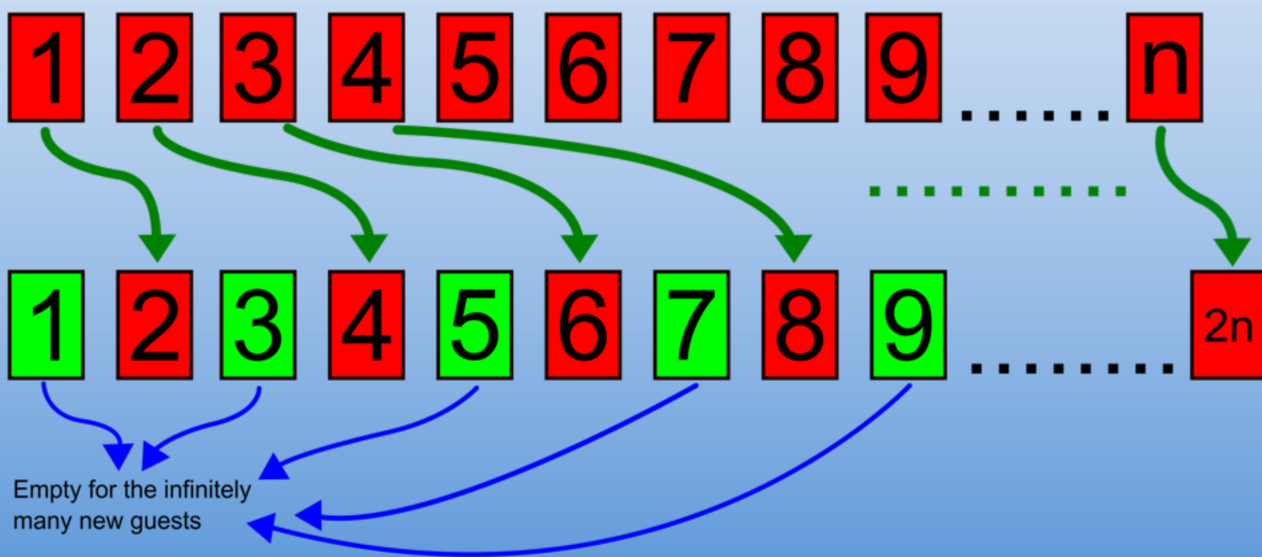
- **Now the first room is empty**, and we can move in.

- **Infinity plus one is still infinity.**

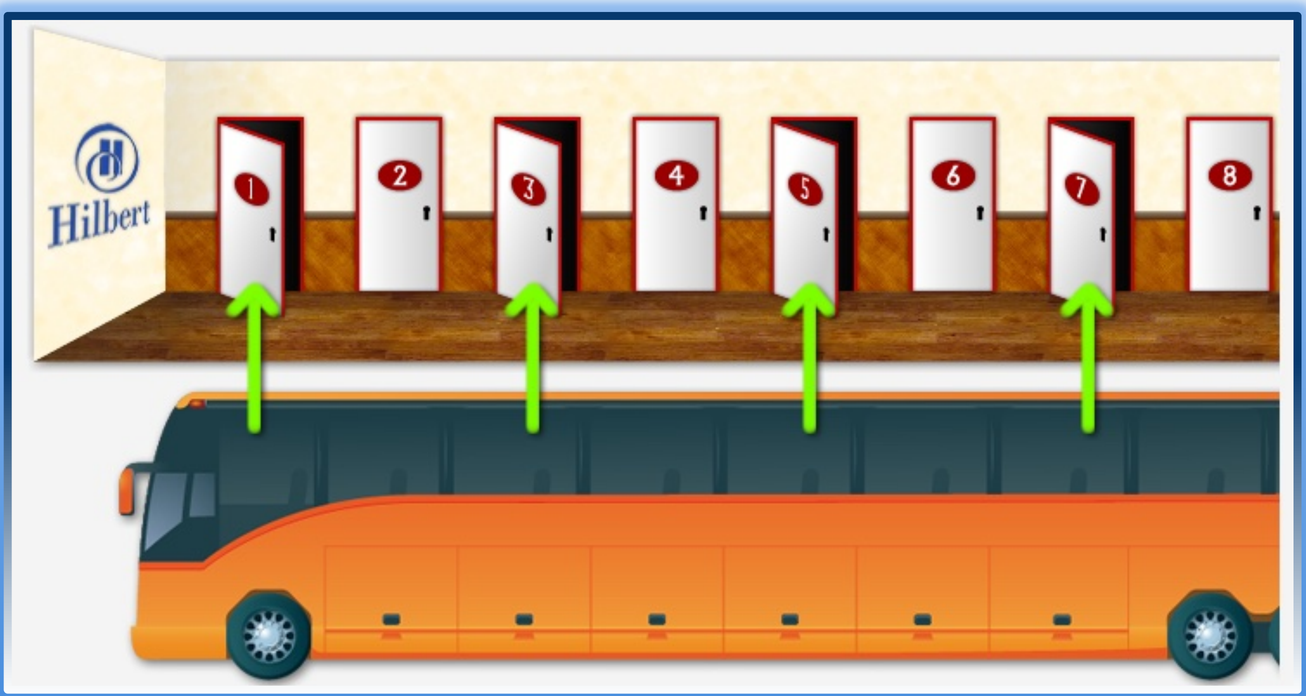
- **This idea can be extended: If 10 new guests arrive**, the reception just asks all guests to move up 10 rooms. Thus even if the hotel is **full** it can still **offer rooms to any number of new guests**!

- Let us suppose that the hotel is **full** again, and that **infinitely many new guests arrive**. Now **the previous method won't work anymore:** you can't ask the guests to move up infinitely many rooms – they would never arrive in their new room.

- Instead **the hotel does something very clever**: they ask their guests to move into the **room with twice the number of their current room.** The guest in room n will move into room 2n.



- Notice that now all **the odd numbered rooms are empty.** Since there are infinitely many odd numbers, there is enough space for the infinitely many new guests.



- **Infinity plus infinity is still infinity.**

- Finally let us suppose that **a whole bus company arrives at the hotel**: they have **infinitely many buses, each with infinitely many seats.**

- **Can we also move all those infinitely times infinitely many passengers into one hotel?**

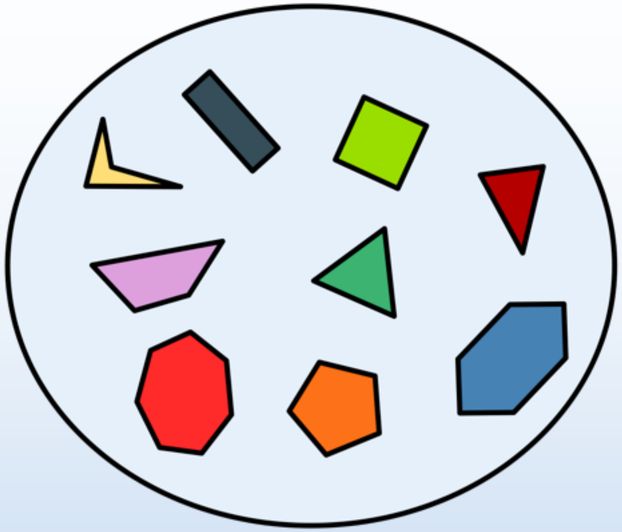
- It turns out that **we can**, using a simple rule: the passenger sitting on seat number i in bus number j gets room $2^i \cdot 3^j$ in the hotel.

- We can be sure not to give two passengers the same room.

- This is because 2 and 3 are prime numbers and because **prime factorisation** is unique.

- In addition, **all rooms which have prime factors other than 2 and 3 remain empty**. As we walk down the infinite hotel corridor, only very few of the rooms will be used. This is useful in case an infinite bus arrives.

COUNTABILITY



- If we want to go further, we have to develop a **new language** to express our ideas. This language is called **Set Theory**.
- A **set** is simply a collection of any objects.
- We say that **two sets have the same size** if we can pair up their elements so that no element is left out or connected to two elements. Such a pairing is called a **bijection**.
- This applies to finite sets as well as infinite sets, and we say that a set is **countable** if it can **be paired up with the natural numbers** $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$.

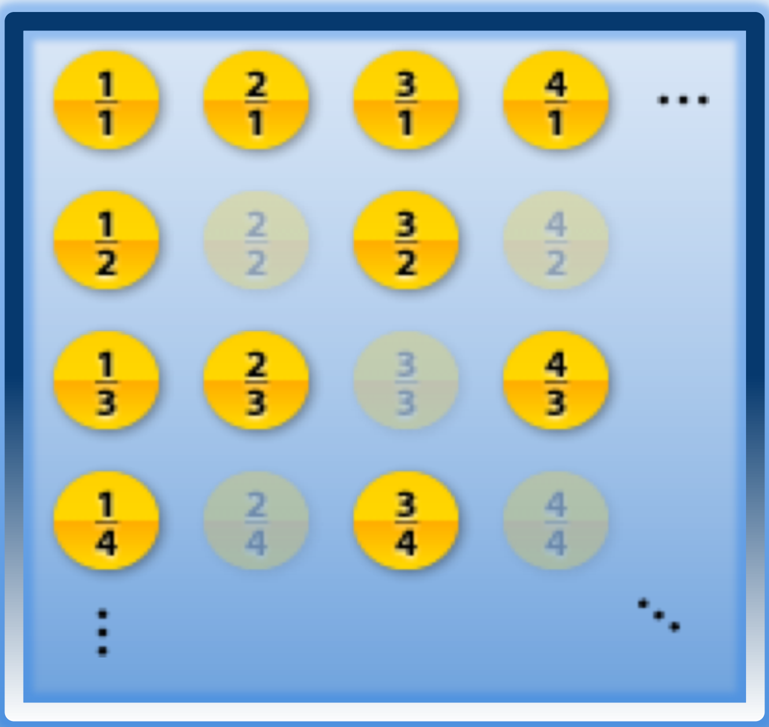
- For example, we can add one more number to $\mathbb{N}$, say 0, and see whether we can pair up the two sets:

- Since the black and the yellow set can be paired up without gaps or overlaps, they must have the **same size**. **Infinity plus one is still infinity.**

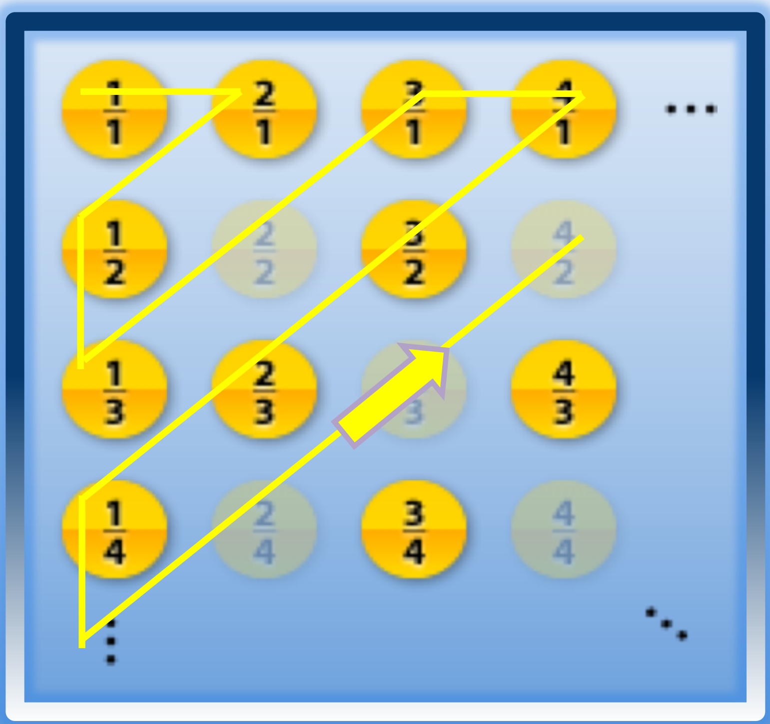
- We can do even **better**: let us try to pair up the **natural numbers** with all **integers** $\mathbb{Z}$ = {…,-3 ,-2, -1, 0, 1, 2, 3, …}. **Jumping back and forth** between the positive and negative ends of the number line we can be sure, counting the integers, not to forget any element.

- **Infinity plus infinity is still infinity**, and it seems that there are **as many positive integers as there are integers** in total.

- We can even try to pair up the **natural numbers** with the positive **rational numbers**.
- To do this, we first **write down all fractions in an infinite grid** where the **columns** are the **numerators** and **rows** the **denominator**. Some fractions appear several times.
- Therefore we **delete all the duplicates** and only keep the smallest version:

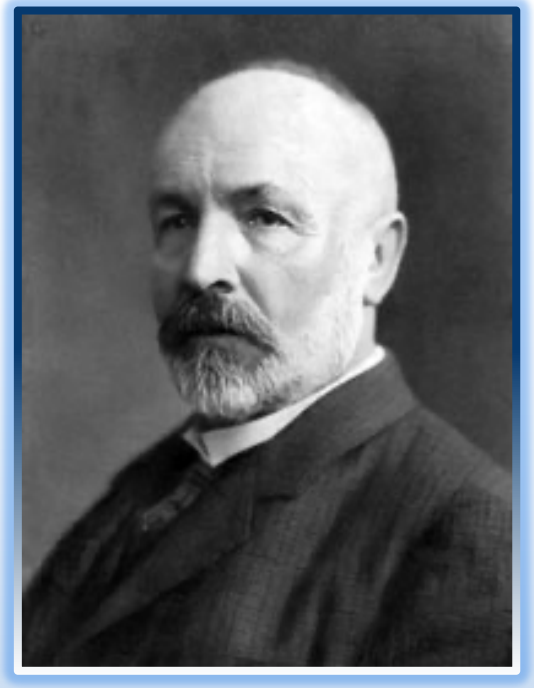


- Now we follow a **path** through this square, and every time we see a valid rational number, we pair it up with the next integer: once again, we can be **sure not to forget any rational number**.



- And this is really quite **surprising**: remember that i.e. there are infinitely many rational numbers between 0 and 1.
- But since we can pair up all these rationals with the natural numbers, there must be the same number. **Infinity times infinity is still infinity.**
- To deduce that **all rational numbers** $\mathbb{Q}$ can be paired up with natural numbers we can follow the same process used to pair up natural numbers and integers.

CANTOR'S DIAGONAL



- **It now seems** quite likely that we must be **able to pair up any number set with the natural numbers**, but **this is not the case.**
- One example are the real numbers $\mathbb{R}$.
- Of course there are infinitely many real numbers, but **this infinity is of a "bigger kind"** than before, and we say that the **real numbers** are **uncountable**.
- To **prove** that the real numbers can't be paired up with the natural numbers, we use **proof by contradiction**.
- We start by assuming that the real numbers were countable. Then we deduce something we know is wrong: a contradiction. This means that our initial assumptions must have been wrong, i.e. that the **real numbers are not countable**.

- All real numbers can be expressed as a **decimal**. The decimal expansion could go on forever or stop, in which case we can add 0s.

- **Let us suppose that the real numbers were countable**. Then we could **write down all real numbers in a list**.

- Below, **every row is a new number**, and the columns are the decimal expansions of the numbers:

x_1	.	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	...
x_2	.	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	...
x_3	.	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	...
x_4	.	x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	...
...							

- We now construct a **new number** consisting of **all the elements along the diagonal:**

$$0\ .\ x_{11}\ x_{22}\ x_{33}\ x_{44}\ x_{55}\ \ldots$$

- Now **we change all the digits** in this new number. For example, we could make every digit a 1, except the 1s, which become 2s:

$$0 \quad . \quad y_{11} \quad y_{22} \quad y_{33} \quad y_{44} \quad y_{55} \quad \ldots$$

- This is again a **real number**.

- Notice, however, that **it is not in our list of all real numbers** above: it can't be the first number, since the $\boldsymbol{x_{11}}$ digit doesn't match. It can't be the second number since the $\boldsymbol{x_{22}}$ digit doesn't match. And so on: $\boldsymbol{0.y_{11}y_{22}y_{33}\ldots}$ is a **completely new real number** not in our initial list.

- But this is a **contradiction**, since we started by assuming that all real numbers were in our list.

- Therefore this assumption must have been wrong. Therefore **the real numbers are not countable**. They are a **bigger kind of infinity**.

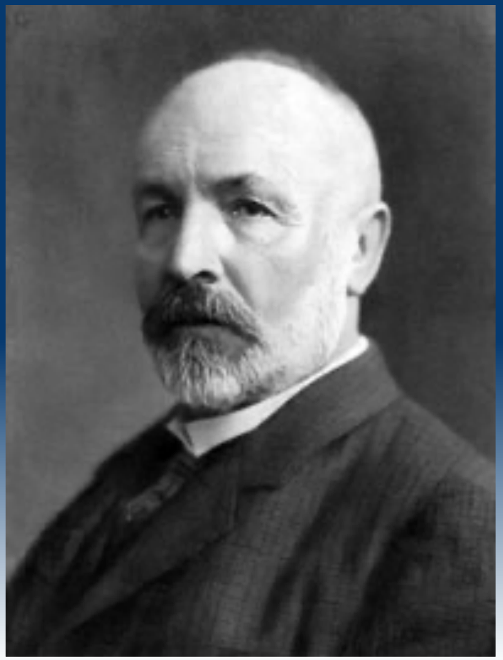
THE CONTINUUM HYPOTHESIS

- **Georg Cantor** (1845–1918) was the first mathematician to realise that there are **different kinds, different sizes of infinity**.

- He called these different sizes **cardinalities.**

- Cantor believed that his study of infinity could explain the absolute infinity of **God**. During his life, his research was dismissed as '**shocking**' and '**corrupting**', but it is **fundamental to modern mathematical logic**.

- We have shown above that $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ all have the **same cardinality**, which is called **Aleph 0** and written as $\aleph_0$ (Aleph is the Hebrew letter A). We have also shown that the real numbers $\mathbb{R}$ have a **bigger cardinality**.



- One question which Georg Cantor asked is whether the cardinality of $\mathbb{R}$ is the next bigger cardinality after $\aleph_0$.

- This is called the **Continuum Hypothesis.**

- Cantor spent many years of his life trying to prove it but **unsuccessfully.**

- Only **much later,** in 1963, mathematicians discovered something very curious: **The continuum hypothesis could be true or false, and it is impossible to prove it either way.** It basically means that you can decide for yourself whether you want it to be true or not.

- The existence of unprovable statements like the **continuum hypothesis** was discovered by **Kurt Gödel.**

UN VIDEO

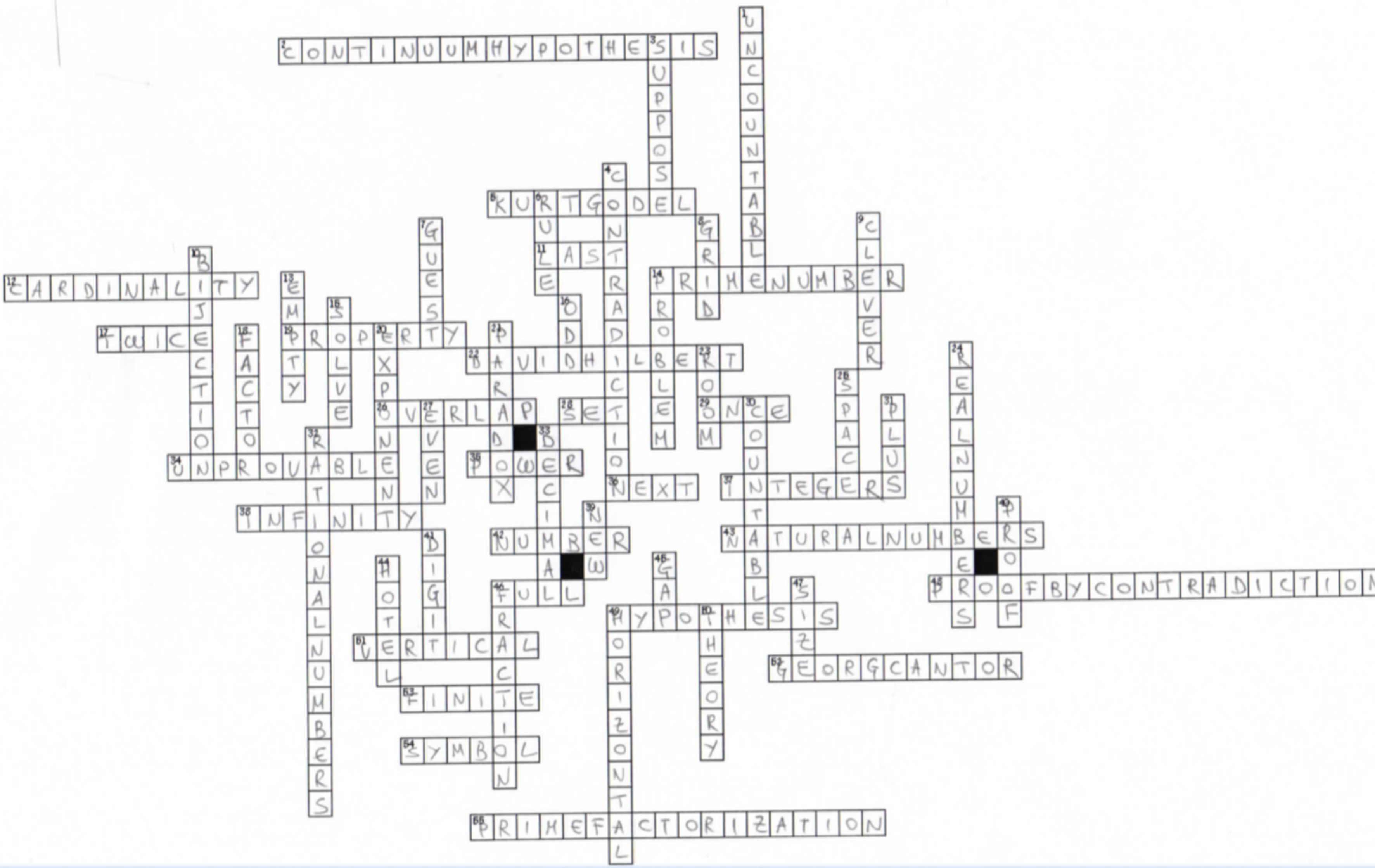
ACTIVITIES

- CROSSWORDS
- FILL IN THE GAPS
- CONCEPTUAL MAP

ANSWER KEY

CROSSWORDS

ANSWERS



Fill in the gaps with the words listed below: (pay attention: not all the words are necessary!)

cardinality - integers – passengers - hotel – **Set Theory** – seats - kind - real numbers – math - empty - guests – paired up - **countable** – size - **Continuum Hypothesis** – full - numbers – mathematician – room –move –cardinality - infinity

Infinity is without doubt one of the hardest mathematical concepts to understand.

The mathematician David Hilbert tried to explain some of the properties of INFINITY using a hotel with infinitely many rooms. We could imagine that it consists of just a single, never-ending corridor, with the rooms numbered 1, 2, 3, 4 and so on. On the day we arrive at this HOTEL, every single room is full. In any other hotel, the reception would have to turn us away.

Here, instead, once we ask for a ROOM, the hotel asks all the guests to move up into the next room. Since there is no last room in the hotel, every guest will get a new room.

Now the first room is EMPTY, and we can move in. The Hilbert doesn't have a last room!

This idea can be extended: thus even if the hotel is FULL it can still offer rooms to *any* finite number of new guests!

Let us suppose that the hotel is full again, and that *infinitely* many new guests arrive. Now the previous method won't work anymore: you can't ask the GUESTS to move up infinitely many rooms. Instead the hotel asks the guests to move into the room with *twice* the number of their current room.

Finally let us suppose that a whole bus company arrives at the hotel: they have infinitely many buses, each with infinitely many SEATS. Even if it's more difficult, we can also move all those infinitely times infinitely many passengers into this already full hotel.

If we want to go even further, we have to develop a new language that is called **SET THEORY**.

A set is simply a collection of any objects. We say that two sets have the same size if we can build a **bijection** pairing up their elements so that no element is left out or connected to two elements. This applies to finite sets as well as infinite sets, and we say that a set is **COUNTABLE** if it can be paired up with the natural numbers.

We can pair up the natural numbers with *all* integers: it seems that there are as many positive INTEGERS as there are integers in total.

We can even pair up the natural NUMBERS with the positive rational numbers.

It now seems quite likely that we must be able to pair up any number set with the natural numbers, but this is not the case. One example are the real numbers. Of course there are infinitely many real numbers, but this infinity is of a "bigger KIND" than before, and we say that the real numbers are *uncountable*. To prove that the real numbers can't be PAIRED UP with the natural numbers we use **proof by contradiction**.

Georg Cantor was the first MATHEMATICIAN to realize that there are different *sizes* of infinity. He called these different sizes **cardinalities**.

We have shown above that $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ all have the same cardinality, which is called $\aleph_0$. We have also shown that the real numbers $\mathbb{R}$ have a bigger CARDINALITY.

Georg Cantor asked whether the cardinality of $\mathbb{R}$ is the next bigger cardinality after $\aleph_0$. This is called the **CONTINUUM HYPOTHESIS**.

FILL IN THE GAPS

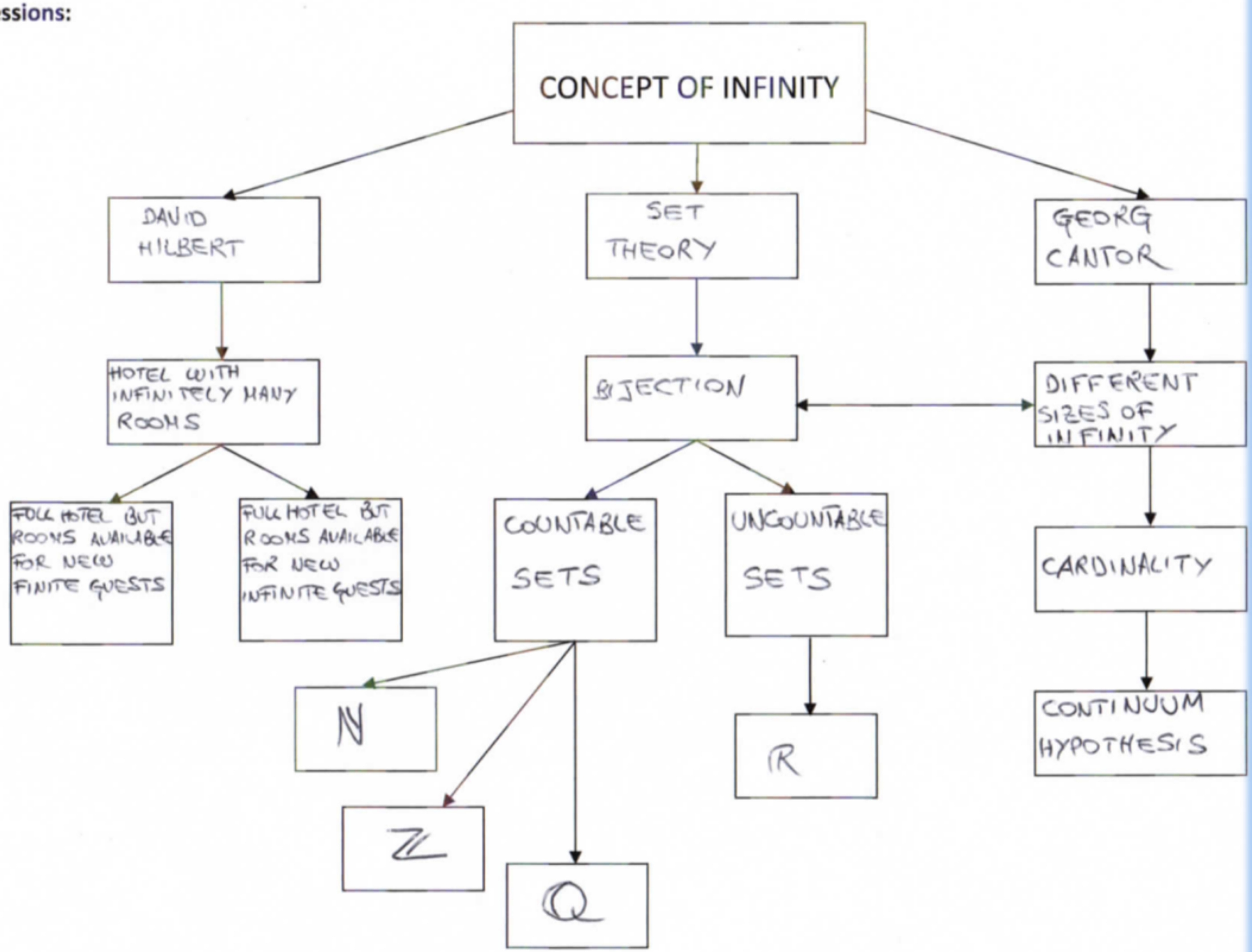
ANSWERS

CONCEPTUAL MAP

ANSWERS

Fill in the blocks with suitable expressions:

- Set Theory
- Countable sets
- Rational numbers ($\mathbb{Q}$)
- Real numbers ($\mathbb{R}$)
- Cardinality
- Georg Cantor
- Full hotel but rooms available for new infinite guests
- Bijection
- Integers ($\mathbb{Z}$)
- Different sizes of infinity
- David Hilbert
- Continuun Hypothesis
- Uncountable sets
- Natural numbers ($\mathbb{N}$)
- Hotel with infinitely many rooms
- Full hotel but rooms available for new finite guests



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